

REGULARITY OF TENSOR PRODUCTS OF k -ALGEBRAS ^(*)S. BOUCHIBA AND S. KABBAJ ⁽¹⁾

ABSTRACT. This paper tackles a problem on the possible transfer of regularity to tensor products of algebras over a field k . If K is a separable extension field of k and A an arbitrary k -algebra, we prove that $K \otimes_k A$ is regular if and only if A is regular and $K \otimes_k A$ is Noetherian. As a consequence, we establish necessary and sufficient conditions for a Noetherian tensor product of two extension fields of k to inherit regularity (in various settings of separability). Throughout, several original examples are provided to illustrate or delimit the scope of the established results.

1. INTRODUCTION

All algebras considered are commutative with identity elements and, unless otherwise specified, are assumed to be non-trivial. All ring homomorphisms are unital. Throughout, k stands for a field. A Noetherian local ring (R, \mathfrak{m}) is regular if its Krull and embedding dimensions coincide; i.e., $\dim(R) = \text{embdim}(R)$, where $\text{embdim}(R)$ denotes the dimension of $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ as an $\frac{R}{\mathfrak{m}}$ -vector space. Regular local rings were first introduced by Krull, and then became prominent once Zariski showed that, geometrically, a regular local ring corresponds to a smooth point on an algebraic variety. Later, Serre found a homological characterization for a local ring R to be regular; that is, R has finite global dimension. Finite global dimension is preserved under localization, so that localizations of regular local rings at prime ideals are again regular. Geometrically, this corresponds to the intuition that if a surface contains a smooth curve, then the surface is smooth near the curve. Consequently, the definition of regularity got globalized as follows: A Noetherian ring R is regular if its localizations with respect to all prime ideals are regular. Using homological techniques, Auslander and Buchsbaum proved in 1950's that every regular local ring is a UFD.

A Noetherian local ring (R, \mathfrak{m}) is a complete intersection if the completion \hat{R} of R with respect to the \mathfrak{m} -adic topology is the quotient ring of a regular local ring modulo an ideal generated by a regular sequence. The ring R is Gorenstein if its injective dimension (as an R -module) is finite; and R is Cohen-Macaulay if grade and height coincide for every ideal of R . These notions are globalized by carrying over to localizations with respect to the prime ideals. We have the following diagram of implications:

$$\begin{array}{c} \text{Regular ring} \\ \Downarrow \\ \text{(Locally) Complete Intersection ring} \end{array}$$

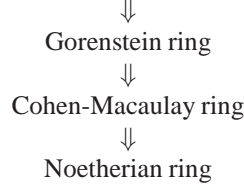
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In this paper we will tackle a problem, originally initiated by Grothendieck [7], on the possible transfer of regularity to tensor products of k -algebras. Recently, it has been proved that a Noetherian tensor product of k -algebras $A \otimes_k B$ inherits from A and B the notions of locally complete intersection ring, Gorenstein ring, and Cohen-Macaulay ring [4, 8, 13, 15, 16]. In particular, $K \otimes_k L$ is a locally complete intersection ring, for any two extension fields K and L of k such that $K \otimes_k L$ is Noetherian [16, Proposition 5]. Notice at this point that tensor products of rings subject to the above concepts were recently used to broaden or delimit the context of validity of some homological conjectures; see for instance [9, 10].

As to regularity, the problem remains elusively open. Indeed, contrary to the above notions, a Noetherian tensor product of two extension fields of k is not regular in general. In 1965, Grothendieck proved that $K \otimes_k L$ is a regular ring provided K or L is a finitely generated separable extension field of k [7, Lemma 6.7.4.1]. In 1969, Watanabe, Ishikawa, Tachibana, and Otsuka, showed that under a suitable condition tensor products of regular rings are complete intersections [19, Theorem 2, p. 417]. In 2003, Tousi and Yassemi proved that a Noetherian tensor product of two k -algebras A and B is regular if and only if so are A and B in the special case where k is perfect; i.e., every (algebraic) extension of k is separable [16, 8].

This paper studies the transfer of regularity in Noetherian tensor products of k -algebras. We first investigate constructions of the form $K \otimes_k A$ where K is a separable extension field of k and A is a k -algebra (Theorem 2.1) and hence generalize Grothendieck's aforementioned result. Then we establish necessary and sufficient conditions for a Noetherian tensor product of two extension fields of k (Theorem 2.4) to inherit regularity. We close with a discussion of the correlation between $A \otimes_k B$ and its fiber rings when subject to regularity. It turns out that, in case A (or B) is assumed to be residually separable, $A \otimes_k B$ is regular if and only if so are A and B (Theorem 2.11). This is a slight improvement of [16, Theorem 6(c)]. All along the paper, several original examples are provided to illustrate or delimit the scope of the established results.

2. TRANSFER OF REGULARITY TO TENSOR PRODUCTS OF k -ALGEBRAS

Our first main result deals with tensor products of the form $K \otimes_k A$ where K is a (not necessarily algebraic) separable extension field of k and A an arbitrary k -algebra. Then we establish necessary and sufficient conditions for a Noetherian tensor product of two extension fields to inherit regularity. We close with a discussion of the transfer of regularity between $A \otimes_k B$ and its fiber rings.

A transcendence base B of an extension field K over k is called a separating transcendence base if K is separable algebraic over $k(B)$; and K is said to be separable over k if every finitely generated intermediate field has a separating transcendence base over k . Finally, recall that a homomorphism $\varphi: A \rightarrow B$ of Noetherian rings is regular if φ is flat and $B \otimes_A \kappa_A(P)$ is regular for each $P \in \text{Spec}(A)$, where $\kappa_A(P)$ denotes the residue field of A_P [12, §32, p. 255].

In 1965, Grothendieck proved that if K and L are two extension fields of k such that either K or L is finitely generated over k and if K is separable over k , then $K \otimes_k L$ is regular [7, Lemma 6.7.4.1]. Next, we generalize this result as well as [16, Proposition 5(b)].

Theorem 2.1. *Let K be a separable extension field of k and A a k -algebra. Then the following assertions are equivalent:*

- (1) $K \otimes_k A$ is regular;
- (2) A is regular and $K \otimes_k A$ is Noetherian.

Proof. The implication (i) \Rightarrow (ii) is straightforward by [12, Theorem 23.7] via localization. To prove (ii) \Rightarrow (i), assume that A is regular and $K \otimes_k A$ is Noetherian.

Step 1. Suppose K is algebraic over k and let $P \in \text{Spec}(A)$. Observe that since the ring $(K \otimes_k A) \otimes_A k_A(P) \cong K \otimes_k k_A(P)$ is reduced [20, Chap. III, §15, Theorem 39] and has Krull dimension zero [14, Theorem 3.1], then it is (von Neumann) regular [11, Ex. 22, p. 64]. This fact makes the canonical faithfully flat homomorphism $\varphi : A \rightarrow K \otimes_k A$ regular. By [12, Theorem 32.2(i)], $K \otimes_k A$ is regular.

Step 2. Suppose K is finitely generated over k . By Mac Lane's Criterion, let $B := \{x_1, \dots, x_t\}$ be a separating transcendence base of K over k and $S := k[B] \setminus \{0\}$. Then $k(B) \otimes_k A \cong S^{-1}A[x_1, \dots, x_t]$ is regular and K is a separable algebraic extension of $k(B)$. By Step 1, $K \otimes_k A \cong K \otimes_{k(B)} (k(B) \otimes_k A)$ is regular.

Step 3. Suppose K is not finitely generated over k . Let Δ denote the set of all finitely generated extension fields of k contained in K and let

$$D := K \otimes_k A = \varinjlim_{E \in \Delta} D(E).$$

where $D(E) := E \otimes_k A$. Let P be a prime ideal of D . The same arguments as in the proof of [3, Proposition 4.14] yield

$$P = (P \cap D(E))D$$

for some $E \in \Delta$ and where the needed Locally Finite Dimensional assumption is ensured here by Noetherianity. Next, set $P(E) := P \cap D(E)$. We have

$$PD_P = (P(E)D(E)_{P(E)})D_P.$$

By Step 2, $D(E)$ is regular. Then, there exists a $D(E)_{P(E)}$ -regular sequence x_1, \dots, x_r of $P(E)D(E)_{P(E)}$ with $P(E)D(E)_{P(E)} = (x_1, \dots, x_r)D(E)_{P(E)}$. Since

$$D_P = (K \otimes_k A)_P \cong (K \otimes_E D(E)_{P(E)})_P,$$

then $1 \otimes x_1, \dots, 1 \otimes x_r$ is a D_P -regular sequence of PD_P with

$$PD_P = (1 \otimes x_1, \dots, 1 \otimes x_r)D_P.$$

Hence, D_P is regular and so is D , as desired. \square

Example 2.12 shows that this theorem is not true, in general, if one substitutes pure inseparability for separability; and that, however, this latter condition is not necessary.

Recall that if K and L are two extension fields of k such that one of them is finitely generated, then $K \otimes_k L$ is Noetherian [19]. The converse is not true in general; e.g.,

$$\mathbb{Q}(x_1, x_2, \dots) \otimes \mathbb{Q}(\sqrt{2}, \sqrt{3}, \dots) \cong \mathbb{Q}(\sqrt{2}, \sqrt{3}, \dots)(x_1, x_2, \dots)$$

is a field, where x_1, x_2, \dots are infinitely many indeterminates over \mathbb{Q} . However, the converse holds in the case $K = L$ [17, Theorem 11]. These facts combined with Theorem 2.1 yield the following corollary, where the separability assumption is required only for regularity.

Corollary 2.2. *Let K and L be two extension fields of k and assume that K is separable over k . Then:*

K or L is finitely generated $\Rightarrow K \otimes_k L$ is Noetherian $\Leftrightarrow K \otimes_k L$ is regular.

The special case where $K = L$ is handled by Corollary 2.6.

For an arbitrary k -algebra A (not necessarily a domain), the transcendence degree over k is given by (cf. [18, p. 392])

$$\text{t.d.}(A : k) := \sup\{\text{t.d.}\left(\frac{A}{p} : k\right) \mid p \in \text{Spec}(A)\}.$$

Further, if A and B are two k -algebras such that $A \otimes_k B$ is Noetherian, then necessarily A and B are Noetherian rings and either $\text{t.d.}(A : k) < \infty$ or $\text{t.d.}(B : k) < \infty$ (cf. [4, p. 69]). Also, for any two extension fields K and L of k , [14, Theorem 3.1] asserts that

$$\dim(K \otimes_k L) = \min\{\text{t.d.}(K : k), \text{t.d.}(L : k)\}.$$

These facts allow one to give illustrative examples (for Corollary 2.2 and hence Theorem 2.1) of regular tensor products (of fields) of arbitrary dimension.

Example 2.3. Let x_1, x_2, \dots be infinitely many indeterminates over k . Then, for any positive integer n , $k(x_1, \dots, x_n) \otimes_k k(x_1, x_2, \dots)$ is an n -dimensional regular ring.

Note that $k(x_1, \dots, x_n)$ and $k(x_1, x_2, \dots)$ are (non-algebraic) separable extensions of k by Mac Lane's Criterion. For the algebraic separable case, see Example 2.9.

Let K and L be two extension fields of k . Assume that K is purely inseparable over k and let \bar{L} be an algebraic closure of L . Then there exists a unique k -homomorphism $u : K \rightarrow \bar{L}$ [5, Proposition 3, p. V.25], and the isomorphic image $u(K)$ is obviously purely inseparable over k . In this vein, we can always view K and L as subfields of a common field \bar{L} . Recall Mac Lane's notion of linear disjointness; namely, K and L are linearly disjoint over k if every subset of K which is linearly independent over k is also linearly independent over L ; equivalently, if $K \otimes_k L$ is a domain.

In the sequel, given an extension field K of k , K_s and K_i will denote the (not necessarily algebraic) separable closure and (algebraic) purely inseparable closure of k in K , respectively. Notice that K is an extension field of the composite field $K_s K_i$ and the equality $K_s K_i = K$ holds, for instance, when K is separable, purely inseparable, or normal over k .

The next result handles the tensor products of two extensions fields, which will be used in the general case as well as to generate new and original examples of tensor products of extension fields of k that are regular.

Theorem 2.4. *Let K and L be two extension fields of k such that $K \otimes_k L$ is Noetherian. Assume that $K = K_s K_i$ and let $K_i = k(S)$ for some generating subset S of K_i . Then the following assertions are equivalent:*

- (1) $K \otimes_k L$ is regular;
- (2) $K_i \otimes_k L$ is a domain;
- (3) $K_i \otimes_k L$ is a field;
- (4) $[k(S') : k] = [L(S') : L]$ for each finite subset S' of S ;
- (5) $K_i \cap L(S') = k(S')$ for each finite subset S' of S .

Proof. Let $p := \text{char}(k)$. The theorem easily holds when $p = 0$ (in which case k is perfect). Next, assume $p \geq 1$. Since K_s is a separable extension of k , $K_s \otimes_k K_i$ is reduced [20, Chap. III, §15, Theorem 39]. Further, since K_i is algebraic over k , $K_s \otimes_k K_i$ is zero-dimensional [14, Theorem 3.1] and hence a von Neumann regular ring [11, Ex. 22, p. 64]. By [17, Proposition 2(c)], $K_s \otimes_k K_i$ has one unique minimal prime ideal. It follows that $K_s \otimes_k K_i$ is

local and therefore a field. Now, consider the surjective ring homomorphism $\varphi : K_s \otimes_k K_i \rightarrow K_s(K_i)$, given on generators of $K_s \otimes_k K_i$ by $a \otimes b \mapsto ab$ (as K_s and K_i may be contained in a common field). So φ is an isomorphism; that is, $K_s \otimes_k K_i \cong K_s K_i = K$. By Theorem 2.1, $K \otimes_k L \cong K_s \otimes_k (K_i \otimes_k L)$ is regular if and only if $K_i \otimes_k L$ is regular. Hence, for the rest of the proof, we may suppose that K is a purely inseparable algebraic extension field of k (i.e., $K = K_i$) with $\text{char}(k) = p \neq 0$. Same arguments as above yield $K \otimes_k L$ is a zero-dimensional local ring and, therefore, (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i). Moreover, the assumption “ $K \otimes_k L$ is a domain” is equivalent to saying that “ K and L are linearly disjoint over k ,” as mentioned above. So that we get (ii) \Leftrightarrow (iv) by [5, Proposition 5 (a), p. V.13] and (ii) \Rightarrow (v) by [5, p. V.13] and via the isomorphism $K \otimes_k L \cong K \otimes_{k(S')} (k(S') \otimes_k L)$ for each finite subset S' of S .

(v) \Rightarrow (iii) Let $x \in S$ and let $p^m = [k(x) : k]$ with m an integer ≥ 0 . Then $a := x^{p^m} \in k$. We wish to show that $k(x) \otimes_k L$ is a field. We may assume $x \notin k$. By (v), $x^{p^r} \notin K \cap L = k$ for each positive integer $r < m$. Therefore, $x \in \bar{L} \setminus L$, where \bar{L} denotes an algebraic closure of L , forcing $(X^{p^m} - a) = (X^{p^r} - x^{p^r})^{p^{m-r}}$ for each positive integer $r < m$ to be irreducible in $L[X]$. It follows that

$$k(x) \otimes_k L \cong \frac{k[X]}{(X^{p^m} - a)} \otimes_k L \cong \frac{L[X]}{(X^{p^m} - a)} \cong L[x] = L(x)$$

where X denotes an indeterminate over \bar{L} . So $k(x) \otimes_k L$ is a field. Next, let $x_1, \dots, x_n \in S$. We have

$$k(x_1, \dots, x_n) \otimes_k L \cong k(x_1, \dots, x_n) \otimes_{k(x_1, \dots, x_{n-1})} (k(x_1, \dots, x_{n-1}) \otimes_k L).$$

By induction on n , $k(x_1, \dots, x_{n-1}) \otimes_k L \cong L(x_1, \dots, x_{n-1})$ is a field and, by (v), we get

$$k(x_1, \dots, x_n) \cap L(x_1, \dots, x_{n-1}) \subseteq K \cap L(x_1, \dots, x_{n-1}) = k(x_1, \dots, x_{n-1})$$

so that

$$k(x_1, \dots, x_n) \cap L(x_1, \dots, x_{n-1}) = k(x_1, \dots, x_{n-1}).$$

Hence, the first step yields

$$k(x_1, \dots, x_n) \otimes_k L \cong k(x_1, \dots, x_{n-1})(x_n) \otimes_{k(x_1, \dots, x_{n-1})} L(x_1, \dots, x_{n-1})$$

is a field. Let Δ denote the set of all finite subset S' of S and observe that

$$K \otimes_k L = \varinjlim_{S' \in \Delta} k(S') \otimes_k L.$$

Thus, $k(S') \otimes_k L$ is a field, for each $S' \in \Delta$, and so is their direct limit $K \otimes_k L$, establishing (iii) and completing the proof of the theorem. \square

One can use Theorem 2.4(v) to build new examples of regular tensor products of fields, as illustrated by the next example.

Example 2.5. Let p be a prime element of \mathbb{Z} and let $y_1, y_2, \dots, y_m, x_1, x_2, \dots, x_n, \dots$ be indeterminates over $\frac{\mathbb{Z}}{p\mathbb{Z}}$. Let

$$\begin{aligned} k &:= \frac{\mathbb{Z}}{p\mathbb{Z}}(y_1^p, y_2^p, \dots, y_m^p, x_1^p, x_2^p, \dots, x_n^p, \dots), \\ K &:= k(x_1, x_2, \dots, x_n, \dots), \\ L &:= k(y_1, y_2, \dots, y_m). \end{aligned}$$

Then $K \otimes_k L$ is a regular ring.

Indeed, notice that K and L are purely inseparable extension fields of k with $[L : k] < \infty$. Also, we have

$$\begin{aligned} K &= \frac{\mathbb{Z}}{p\mathbb{Z}}(y_1^p, y_2^{p^2}, \dots, y_m^{p^m}, x_1, x_2, \dots, x_n, \dots), \\ L &= \frac{\mathbb{Z}}{p\mathbb{Z}}(y_1, y_2, \dots, y_m, x_1^p, x_2^{p^2}, \dots, x_n^{p^n}, \dots). \end{aligned}$$

Next, let $x_{i_1}, x_{i_2}, \dots, x_{i_r}$ be a finite subset of $\{x_1, x_2, \dots, x_n, \dots\}$. Then

$$\begin{aligned} K \cap L(x_{i_1}, x_{i_2}, \dots, x_{i_r}) &= \frac{\mathbb{Z}}{p\mathbb{Z}}(x_{i_1}, x_{i_2}, \dots, x_{i_r}, x_1^p, x_2^{p^2}, \dots, x_n^{p^n}, \dots, y_1^p, y_2^{p^2}, \dots, y_m^{p^m}) \\ &= k(x_{i_1}, x_{i_2}, \dots, x_{i_r}). \end{aligned}$$

Hence, by Theorem 2.4(v), $K \otimes_k L$ is regular, as desired.

Next, we handle the case where $K = L$ via a combination of Theorem 2.1 and Theorem 2.4(v).

Corollary 2.6. *Let K be an extension field of k . Then the following assertions are equivalent:*

- (1) $K \otimes_k K$ is regular;
- (2) $K \otimes_k K$ is Noetherian and K is separable over k ;
- (3) K is a finitely generated separable extension field of k .

Proof. (i) \Rightarrow (ii) Assume that $K \otimes_k K$ is regular. Then $K \otimes_k K$ is Noetherian, so that K is finitely generated over k . We claim that $K \otimes_E K$ is regular for any extension field E of k contained in K . In effect, let E be a field extension of k contained in K . Then

$$\begin{aligned} K \otimes_k K &\cong K \otimes_E (E \otimes_k K) \\ &\cong K \otimes_E (K \otimes_k E) \\ &\cong (K \otimes_E K) \otimes_k E \quad (\text{Cf. [1, Ex. 2.15, p. 27]}). \end{aligned}$$

It follows, by [12, Theorem 23.7] and by localization, that $K \otimes_E K$ is regular, establishing the claim. Now, let B be a finite transcendence basis of K over k and let E be the algebraic separable closure of $k(B)$ in K . Then, via the above claim, $K \otimes_E K$ is regular and K is purely inseparable over E . By Theorem 2.4(v), $K = E$. It follows that K is separable over k , as desired.

(ii) \Rightarrow (iii) is handled by [17, Theorem 11] (as mentioned before) and (iii) \Rightarrow (i) follows easily from Theorem 2.1, completing the proof of the corollary. \square

One can use Theorem 2.4(v) or Corollary 2.6 to build (zero-dimensional Noetherian local) tensor products of fields that are complete intersection but not regular, as shown below.

Example 2.7. Let $k \subsetneq K \subseteq L$ be extension fields such that K is purely inseparable over k and $K \otimes_k L$ is Noetherian. Then $K \otimes_k L$ is a complete intersection ring [16, Proposition 5(a)] which is not regular by Theorem 2.4(v) (or Corollary 2.6). For instance, for any prime p , one may simply take

$$k := \frac{\mathbb{Z}}{p\mathbb{Z}}(x^p) \text{ and } K = L := \frac{\mathbb{Z}}{p\mathbb{Z}}(x)$$

where x is an indeterminate over $\frac{\mathbb{Z}}{p\mathbb{Z}}$.

The next result handles the (algebraic) separable case featuring a slight generalization of [17, Proposition 8]. Recall, for convenience, that if K is a separable extension of k , then $K \otimes_k L$ is always reduced for any extension field L of k [20, Chap. III, §15, Theorem 39].

Corollary 2.8. *Let K and L be two extension fields of k such that $K \otimes_k L$ is Noetherian. Assume that K is algebraic over k . Then the following assertions are equivalent:*

- (1) $K \otimes_k L$ is (von Neumann) regular;
- (2) $K \otimes_k L$ is reduced;
- (3) $K \otimes_k L$ is a finite product of fields.

If, in addition, K is separable and L is Galois over k such that K, L are contained in an algebraic closure of k , then the above are equivalent to:

- (iv) $n := [K \cap L : k] < \infty$.

Moreover, $K \otimes_k L$ is isomorphic to the product of n copies of the field $K(L)$.

Proof. By [14, Theorem 3.1], $\dim(K \otimes_k L) = 0$. Recall at this point that a zero-dimensional Noetherian ring is regular if and only if it is von Neumann regular. So a combination of [11, Theorem 164], [11, Ex. 22, p. 64], and [17, Lemma 0] yields (i) \Leftrightarrow (ii) \Leftrightarrow (iii). The last two statements are handled by [17, Proposition 8]. \square

Next, we give an illustrative example for this corollary.

Example 2.9. Let $(p_j)_{j \geq 1}$ denote the sequence of all prime numbers. Let

$$X := \{i, e^{\frac{2i\pi}{3}}\} \cup \{\sqrt{p_j} \mid j \text{ odd}\} \text{ and } Y := \{i\} \cup \{\sqrt{p_j} \mid j \text{ even}\}.$$

Clearly, $\mathbb{Q}(X)$ (resp., $\mathbb{Q}(Y)$) is an infinite algebraic separable non-normal (resp., Galois) extension field of \mathbb{Q} and hence by Corollary 2.8

$$\mathbb{Q}(X) \otimes \mathbb{Q}(Y) \cong \mathbb{Q}(i, e^{\frac{2i\pi}{3}}, \sqrt{2}, \sqrt{3}, \dots) \times \mathbb{Q}(i, e^{\frac{2i\pi}{3}}, \sqrt{2}, \sqrt{3}, \dots)$$

is a non-trivial zero-dimensional regular ring.

Next, we move to the general case, where we discuss the correlation between $A \otimes_k B$ and its fiber rings when subject to regularity. Let A and B be two k -algebras. By identifying A and B with their canonical images in $A \otimes_k B$, one can view $A \otimes_k B$ as a free (hence faithfully flat) extension of A and B . This very fact lies behind the known transfers of regularity between $A \otimes_k B$ and its fiber rings over the prime ideals of A or B . The next result collects these transfer results along with a slight generalization of [16, Theorem 6(c)]. We also provide an example, via Theorem 2.4, for the non-reversibility in general of the implications involved. For this purpose, we first make the following definition.

Definition 2.10. A k -algebra R is said to be residually separable, if $\kappa_R(P)$ is separable over k for each $P \in \text{Spec}(R)$, where $\kappa_R(P)$ denotes the residue field of R_P .

It is easily seen that a field k is perfect if and only if every k -algebra is residually separable. More examples of residually separable k -algebras are readily available through localizations of polynomial rings or pullback constructions [2, 6]. For instance, let x be an indeterminate over k and $K \subseteq L$ two separable extension fields of k . Let

$$R := L[x]_{(x)} \text{ and } S := K + xL[x]_{(x)}.$$

Note that the extensions

$$k \subseteq K \subseteq L \subseteq L(x) = \text{qf}(R) = \text{qf}(S)$$

are separable by Mac Lane's Criterion and transitivity of separability. So that R and S are residually separable k -algebras.

Theorem 2.11. *Let A and B be two k -algebras such that $A \otimes_k B$ is Noetherian. Consider the following assertions:*

- (1) A , B , and $\kappa_A(P) \otimes_k \kappa_B(Q)$ are regular $\forall (P, Q) \in \text{Spec}(A) \times \text{Spec}(B)$;
- (2) B and $A \otimes_k \kappa_B(Q)$ are regular $\forall Q \in \text{Spec}(B)$;
- (3) A and $\kappa_A(P) \otimes_k B$ are regular $\forall P \in \text{Spec}(A)$;
- (4) $A \otimes_k B$ is regular;
- (5) A and B are regular.

Then (i) \Rightarrow (ii) (resp., (iii)) \Rightarrow (iv) \Rightarrow (v). If A (or B) is residually separable, then all assertions are equivalent.

Proof. The first statement is a combination of Corollary 2 and Corollary 4 as well as the proof of Theorem 6 in [16].

Next, suppose that A or B is residually separable. Then $\kappa_A(P) \otimes_k \kappa_B(Q)$ is always regular by Theorem 2.1 for any $P \in \text{Spec}(A)$ and $Q \in \text{Spec}(B)$; and, hence, so are $\kappa_A(P) \otimes_k B$ and $A \otimes_k \kappa_B(Q)$. Moreover, recall that Noetherianity carries over to $\kappa_A(P) \otimes_k \kappa_B(Q)$ via localization of the general fact that if I and J are proper ideals of A and B , respectively, then

$$\frac{A \otimes_k B}{I \otimes_k B + A \otimes_k J} \cong \frac{A}{I} \otimes_k \frac{B}{J}.$$

Thus, the five assertions in the theorem collapse to: “ $A \otimes_k B$ is regular if and only if A and B are regular.” \square

The above implications are not reversible in general, as shown by the next example. This example shows also that the separable assumption in Theorem 2.1 is sufficient but not necessary and it does not hold, in general, for purely inseparable extensions.

Example 2.12. Let K be a purely inseparable extension field of k with $\text{char}(k) = p \neq 0$ and let $u \in K$ with $p^e := [k(u) : k]$ for some $e \geq 2$. Then $a := u^{p^e} \in k$. Let x be an indeterminate over k , $r \in \{1, \dots, e-1\}$, and $A := k[x]_{(x^{p^{e-r}} - a)}$. Then:

- (1) A is local regular with maximal ideal $\mathfrak{m} := (x^{p^{e-r}} - a)A$.
- (2) $k(u) \otimes_k A$ is regular.
- (3) $k(u) \otimes_k \frac{A}{\mathfrak{m}}$ is not regular.

Indeed, notice that $(x^{p^{e-r}} - a)$ is a prime ideal of $k[x]$ and, hence, \mathfrak{m} is the maximal ideal of A , since $\frac{A}{\mathfrak{m}} \cong \frac{k[x]}{(x^{p^{e-r}} - a)} \cong k(u^{p^r})$. Moreover, $k(u) \otimes_k A \cong S^{-1}k(u)[x]$ is a regular ring, where $S := k[x] \setminus (x^{p^{e-r}} - a)$. This proves (i) and (ii). However, $k(u) \otimes_k \frac{A}{\mathfrak{m}} \cong k(u) \otimes_k k(u^{p^r})$ is not regular, by Theorem 2.4(v), since $k \neq k(u) \cap k(u^{p^r}) = k(u^{p^r})$, proving (iii).

The assumption “ A (or B) is residually separable” in Theorem 2.11 is not necessary, as shown by the following example.

Example 2.13. Let k , K , and L be defined as in Example 2.5 and x, y two indeterminates over k . Let

$$\begin{aligned} A &:= K[x]_{(x)} = K + \mathfrak{m}_A \quad \text{with } \mathfrak{m}_A := xA \\ B &:= L[y]_{(y)} = L + \mathfrak{m}_B \quad \text{with } \mathfrak{m}_B := yB \end{aligned}$$

Then A and B are regular local k -algebras which are not residually separable over k (since K and L are purely inseparable over k as seen in Example 2.5). Moreover, $A \otimes_k B$ is

Noetherian (in fact, regular via localization) and $\frac{A}{\mathfrak{m}_A} \otimes_k \frac{B}{\mathfrak{m}_B} \cong K \otimes_k L$ is a regular ring. Consequently, A and B satisfy all assertions of Theorem 2.11, as desired.

The next example illustrates the slight improvement (of [16, Theorem 6(c)]) featured in the last statement of Theorem 2.11. Namely, we provide original examples where k is an arbitrary field, A, B are regular k -algebras with $A \otimes_k B$ Noetherian and A is residually separable over k .

Example 2.14. Let k be an arbitrary field, K any separable extension field of k , and x, y, t three indeterminates over k . Consider the K -algebra homomorphism

$$\varphi : K[x, y] \rightarrow K[[t]]$$

defined by $\varphi(x) = t$ and $\varphi(y) = s := \sum_{n \geq 1} t^{n!}$. Since s is known to be transcendental over $K(t)$, φ is injective. This induces the following embedding of fields

$$\overline{\varphi} : K(x, y) \rightarrow K((t)).$$

It is easy to check that $A := \overline{\varphi}^{-1}(K[[t]])$ is a discrete rank-one valuation overring of $K[x, y]$ and that $A = K + \mathfrak{m}$ with $\mathfrak{m} = xA$. Then, A is a residually separable regular ring. Now, let B be any regular ring such that $A \otimes_k B$ is Noetherian. For instance, one may choose B to be any finitely generated regular k -algebra or any (purely inseparable) finitely generated extension field of k . By Theorem 2.11, $A \otimes_k B$ is a regular ring.

It is worthwhile noticing that, in most examples, the non-regularity was ensured by the negation of “ $K_i \cap L = k$.” One might wonder if this weak property may generate the condition (v) of Theorem 2.4; namely, let K be a finite dimensional purely inseparable extension field of k and let L be an extension field of k . Do we have: $K \cap L = k \Leftrightarrow K \otimes_k L$ regular? The answer is negative as shown by the next example.

Example 2.15. Let x, y, z be three indeterminates over $\frac{\mathbb{Z}}{2\mathbb{Z}}$. Let

$$\begin{aligned} k &:= \frac{\mathbb{Z}}{2\mathbb{Z}}(x^4, y^4), \\ K &:= k(x^2, y^2) = \frac{\mathbb{Z}}{2\mathbb{Z}}(x^2, y^2), \\ L &:= k(x^2(y^2 + z), z) = \frac{\mathbb{Z}}{2\mathbb{Z}}(x^4, x^2(y^2 + z), z). \end{aligned}$$

Then $K \cap L = k$ and $K \otimes_k L$ is not a regular ring.

Indeed, clearly, K is a purely inseparable extension field of k . Further, note that $\{1, x^2\}$ is a basis of K over $k(y^2)$ and, as $(x^2(y^2 + z))^2 \in k(z)$, $\{1, x^2(y^2 + z)\}$ is a basis of L over $k(z)$. Let $f \in K \cap L$. So there exist $g_0, g_1 \in k(y^2)$ and $f_0, f_1 \in k(z)$ such that

$$\begin{cases} f &= g_0 + g_1 x^2 \\ &= f_0 + f_1 x^2(y^2 + z). \end{cases}$$

As $(x^2)^2 \in k(y^2, z)$ and $x^2 \notin k(y^2, z) = \frac{\mathbb{Z}}{2\mathbb{Z}}(x^4, y^2, z)$, then $\{1, x^2\}$ is, as well, a basis of $k(x^2, y^2, z)$ over $k(y^2, z)$. It follows that $f_0 = g_0$ and $f_1(y^2 + z) = g_1$. Hence, $f_0 \in k(z) \cap k(y^2) = k$. Moreover, observe that $\{1, y^2\}$ is a basis of $k(y^2, z)$ over $k(z)$ and of $k(y^2)$ over k . Hence, as $g_1 = f_1 z + f_1 y^2$ and $g_1 \in k(y^2)$, we get $f_1 z \in k$, so that $f_1 = 0$. Consequently, $f \in k$ and therefore $K \cap L = k$, as claimed.

Now, $L(x^2) = k(x^2, y^2, z) = K(z)$. Hence $K \cap L(x^2) = K \neq k(x^2)$. Then, by Theorem 2.4(v), $K \otimes_k L$ is not regular, as desired.

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